

$$L_p[0, 1] \setminus \bigcup_{q>p} L_q[0, 1] \text{ IS SPACEABLE FOR EVERY } p > 0$$

G. BOTELHO, V. V. FÁVARO, D. PELLEGRINO, AND J. B. SEOANE-SEPÚLVEDA

ABSTRACT. In this short note we prove the result stated in the title; that is, for every $p > 0$ there exists an infinite dimensional closed linear subspace of $L_p[0, 1]$ every nonzero element of which does not belong to $\bigcup_{q>p} L_q[0, 1]$. This answers in the positive a question raised in 2010 by R. M. Aron on the spaceability of the above sets (for both, the Banach and quasi-Banach cases). We also complete some recent results from [2] for subsets of sequence spaces.

As it has become an standard notion in the last years, given a topological vector space X , we say that a subset $M \subset X$ is *spaceable* (see [1]) if there exists an infinite dimensional closed linear subspace $Y \subset M \cup \{0\}$. Very recently, it was proved in [2] that, for every $p > 0$, the set $\ell_p \setminus \bigcup_{0<q<p} \ell_q$ is spaceable. As a consequence of a lecture delivered by the second author at an international conference held in Valencia (Spain) in the summer of 2010, R. M. Aron asked the question of whether a similar result to [2, Corollary 1.7] would hold for L_p -spaces. The aim of this note is to answer R. M. Aron's question in the positive by means of a constructive procedure and some classical Real Analysis and Linear Algebra techniques.

Theorem. $L_p[0, 1] \setminus \bigcup_{q>p} L_q[0, 1]$ is spaceable for every $p > 0$.

PROOF. Let us first consider the following representation of the semi-open interval $[0, 1)$ as a disjoint union of intervals:

$$[0, 1) = [0, 1 - 1/2) \cup [1 - 1/2, 1 - 1/4) \cup [1 - 1/4, 1 - 1/8) \cup \cdots = \bigcup_{n=1}^{\infty} I_n,$$

where $I_n := [a_n, b_n) = [1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n})$. Notice that, for every $n \in \mathbb{N}$ and every $x \in I_n$, there is a unique $x_n \in [0, 1)$ such that

$$x = (1 - x_n)a_n + x_nb_n.$$

Now, given $p > 0$, let us fix a function $f \in L_p[0, 1] - \bigcup_{q>p} L_q[0, 1]$, and define a sequence of functions $(f_n)_{n=1}^{\infty}$, with $f_n: [0, 1] \rightarrow \mathbb{R}$, as follows:

$$f_n(x) = \begin{cases} f(x_n) & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases}$$

The geometric idea is to reproduce the graph of f on the interval I_n . By construction, we have that $\|f_n\|_{L_p} \leq \|f\|_{L_p}$ for every $n \in \mathbb{N}$. Also, the functions f_n are linearly

2010 *Mathematics Subject Classification.* 46A45, 46A16, 46B45.

Key words and phrases. lineability, spaceability, L_p spaces, quasi-Banach spaces.

G. Botelho was supported by CNPq Grant 306981/2008-4, V. V. Fávaro by FAPEMIG Grant CEX-APQ-00208-09, D. Pellegrino by CNPq Grant 620108/2008-8 and J. B. Seoane-Sepúlveda by the Spanish Ministry of Science and Innovation, grant MTM2009-07848.

independent (they have disjoint supports) and

$$\text{span}\{f_n : n \in \mathbb{N}\} \subset L_p[0, 1] \setminus \bigcup_{q>p} L_q[0, 1].$$

The latter proves that $L_p[0, 1] \setminus \bigcup_{q>p} L_q[0, 1]$ is \aleph_0 -lineable (as it was seen in [3] for the case $p > 1$). In order to go further, the strategy shall be to define a bounded linear and injective operator $T: F \longrightarrow L_p[0, 1]$, where F is a Banach space, and such that $\overline{T(F)} \cap L_q[0, 1] = \{0\}$ for every $q > p$. This shall prove $L_p[0, 1] - \bigcup_{q>p} L_q[0, 1]$ is spaceable. Indeed, if $(\alpha_j)_{j=1}^\infty \in \ell_s$, where $s = 1$ if $p \geq 1$ and $s = p$ if $0 < p < 1$, we have

$$\sum_{n=1}^\infty \|\alpha_n f_n\|_{L_p}^s = \sum_{n=1}^\infty |\alpha_n|^s \|f_n\|_{L_p}^s \leq \sum_{n=1}^\infty |\alpha_n|^s \|f\|_{L_p}^s = \|f\|_{L_p}^s \|(\alpha_j)_{j=1}^\infty\|_s^s < +\infty.$$

Since $L_p[0, 1]$ is a Banach space for $p > 1$ and a quasi-Banach space for $0 < p < 1$, it follows that $\sum_{n=1}^\infty \alpha_n f_n \in L_p[0, 1]$ and, thus,

$$T: \ell_s \longrightarrow L_p[0, 1], \quad T((\alpha_j)_{j=1}^\infty) = \sum_{n=1}^\infty \alpha_n f_n$$

is a well defined linear operator. Suppose that $\sum_{n=1}^\infty \alpha_n f_n = 0$. Then (given any $k \in \mathbb{N}$) we have that, for all $x \in I_k$,

$$\alpha_k f_k(x) = \sum_{n=1}^\infty \alpha_n f_n(x) = 0$$

and, since each $f_k \neq 0$, it follows that $\alpha_k = 0$. Hence T is injective and $T(\ell_s)$ is a linear subspace of $L_p[0, 1]$. Considering the closure $\overline{T(\ell_s)}$ of $T(\ell_s)$ in $L_p[0, 1]$, let us prove that $\overline{T(\ell_s)} \cap L_q[0, 1] = \{0\}$ for every $q > p$. Indeed, let $g \in \overline{T(\ell_s)} \setminus \{0\}$. There exist sequences $(a_i^{(k)})_{i=1}^\infty \in \ell_s$ ($k \in \mathbb{N}$) such that $g = \lim_{k \rightarrow \infty} T\left(\left(a_i^{(k)}\right)_{i=1}^\infty\right)$ in $L_p[0, 1]$. Thus, we have

$$\lim_{k \rightarrow \infty} \int_0^1 \left| \sum_{n=1}^\infty a_n^{(k)} f_n(x) - g(x) \right|^p dx = 0,$$

which, in particular, implies that

$$\lim_{k \rightarrow \infty} \int_0^{\frac{1}{2}} \left| a_1^{(k)} f_1(x) - g(x) \right|^p dx = 0.$$

Next, and by means of a subsequence (if needed), we obtain that

$$\lim_{k \rightarrow \infty} a_1^{(k)} f_1(x) = g(x) \text{ a.e. } x \in [0, 1/2).$$

Clearly, the set $S = \{x \in [0, \frac{1}{2}) : f_1(x) \neq 0\}$ is not of measure zero. Thus, given $x' \in S$, we have

$$\lim_{k \rightarrow \infty} a_1^{(k)} = \frac{f_1(x')}{g(x')} = \eta \neq 0,$$

concluding that

$$g(x) = \eta f_1(x) \text{ a.e. } x \in [0, 1/2).$$

which implies that $g \notin L_q[0, 1/2]$ (regardless of the $q > p$), finishing the proof. ■

Remark. Let us point out that the previous result is the best possible in terms of dimension, since $L_p[0, 1]$ is a \mathfrak{c} -dimensional linear space (with \mathfrak{c} denoting the continuum). Also, notice that in the above proof, we actually have that $g \notin L_q(I_n)$ regardless of the $q > p$ and $n \in \mathbb{N}$.

REFERENCES

- [1] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on \mathbb{R}* , Proc. Amer. Math. Soc. **133** (2005), no. 3, 795–803.
- [2] G. Botelho, D. Diniz, V. Fávaro, and D. Pellegrino, *Spaceability in Banach and quasi-Banach sequence spaces*, Linear Algebra Appl. **434** (2011), no. 5, 1255–1260.
- [3] G. A. Muñoz-Fernández, N. Palmberg, D. Puglisi, and J. B. Seoane-Sepúlveda, *Lineability in subsets of measure and function spaces*, Linear Algebra Appl. **428** (2008), no. 11-12, 2805–2812.

FACULDADE DE MATEMÁTICA,
UNIVERSIDADE FEDERAL DE UBERLÂNDIA,
38.400-902 UBERLÂNDIA, BRAZIL.
E-mail address: botelho@ufu.br, vvfavaro@gmail.com

DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDADE FEDERAL DA PARAÍBA,
58.051-900 - JOÃO PESSOA, BRAZIL.
E-mail address: pellegrino@pq.cnpq.br

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO,
FACULTAD DE CIENCIAS MATEMÁTICAS,
PLAZA DE CIENCIAS 3,
UNIVERSIDAD COMPLUTENSE DE MADRID,
MADRID, 28040, SPAIN.
E-mail address: jseoane@mat.ucm.es